

Black Hole Entropy from complex Ashtekar variables

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In loop quantum gravity, the number $N_\Gamma(a_H, \gamma)$ of microstates of a black hole for a given discrete geometry Γ depends on the so-called Barbero-Immirzi parameter γ . Using a suitable analytic continuation of γ to complex values, we show that the number $N_\Gamma(a_H, \pm i)$ of microstates behaves as $\exp(a_H/(4\ell_{\text{Pl}}^2))$ for large area a_H in the large spin semiclassical limit. Such a correspondence with the semiclassical Bekenstein-Hawking entropy law points towards an unanticipated and remarkable feature of the original complex Ashtekar variables for quantum gravity.

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Introduction

The Barbero-Immirzi parameter γ was originally introduced [1] as a way to circumvent the problem of imposing the reality constraints in the complex (self-dual) Ashtekar formulation of gravity [2]. Historically, γ appeared as a parameter labeling a family of canonical transformations turning the ADM phase space into the so-called Ashtekar-Barbero phase space, parametrized by a real $\mathfrak{su}(2)$ connection and its conjugate momentum. Later on, Holst [3] realized that this Hamiltonian formulation of gravity could be obtained by adding to the standard Hilbert-Palatini Lagrangian a topological term with γ as a coupling constant. This term vanishes due to the Bianchi identities when one resolves the spin connection in terms of the tetrad, and for this reason γ is not relevant at the classical level.

In the quantum theory however, the Barbero-Immirzi parameter plays a crucial role since the spectrum of the geometric operators is discrete in units of the loop quantum gravity (LQG) scale $\ell_{\text{LQG}} = \sqrt{\gamma G \hbar} = \sqrt{\gamma} \ell_{\text{Pl}}$, where ℓ_{Pl} is the Planck length [4]. Moreover, this γ -dependency of the fundamental physical cut-off is inherited by the value of the black hole entropy in the LQG calculation. Compatibility with the expected semiclassical value $S = a_H/(4\ell_{\text{Pl}}^2)$ (where a_H is the area of the horizon) requires that γ be fixed to a particular real value. In fact, a lot of different techniques have been developed in order to obtain the value of γ [5, 6] (see also [7]).

In LQG, the horizon of a black hole has the topology of a 2-sphere, with colored punctures coming from the spin network links that cross the horizon. Each puncture carries a quantum of area, and the sum of these microscopic areas gives the macroscopic area a_H of the horizon. In the microcanonical ensemble, the entropy of the black hole is given by the logarithm of the number $N(a_H, \gamma)$ of microscopic states compatible with the macroscopic area a_H of the horizon. The quantity $N(a_H, \gamma)$ depends on a_H , but also on γ , since admissible microstates are sets $\Gamma = \{j_1, \dots, j_p\}$ of punctures labelled by spins, where p

is the number of punctures, satisfying the quantum area constraint

$$a_H - \epsilon < 8\pi\gamma\ell_{\text{Pl}}^2 \sum_{\ell=1}^p \sqrt{C(j_\ell)} < a_H + \epsilon, \quad (1)$$

for some small coarse graining $\epsilon > 0$, and $C(j) = j(j+1)$.

It has been shown recently [6] that there is a close relationship between black holes in LQG and $\text{SU}(2)$ Chern-Simons theory, where in particular the level k depends on both a_H and γ . In this framework, the number $N(a_H, \gamma)$ of microstates can be expressed as

$$N(a_H, \gamma) = \sum_{\Gamma} w_{\Gamma} N_{\Gamma}(a_H, \gamma), \quad (2)$$

where $N_{\Gamma}(a_H, \gamma)$ is the dimension of the $\text{SU}(2)$ Chern-Simons theory Hilbert space on a punctured 2-sphere, and w_{Γ} is a weight assigned to each Γ .

So far, the weights w_{Γ} have always been fixed by assuming the equipartition of probability in the space of Γ 's, combined with the assumption of distinguishability of the puncture states. With these assumptions, it is possible to compute the asymptotic behavior of $N(a_H, \gamma)$ for a large area a_H (compared to the Planck area ℓ_{Pl}^2), and to recover the Bekenstein-Hawking entropy with its logarithmic correction, provided that γ is fixed to a particular value. For real values of γ , the exponential growth of the uniform Boltzmann weights w_{Γ} for a large number of punctures is the key property for the recovery of a black hole entropy that is proportional to a_H .

Nevertheless, the present state of development of LQG is not conclusive about the precise form of the weights w_{Γ} , and one must admit that this remains to a large extent an open issue. It seems clear to us that these weights should be fixed by dynamical considerations, in relation in particular to the requirement that a semiclassical geometry plus field configuration (the suitable low energy vacuum state) be recovered near the horizon. In four dimensions, some first steps in this direction have

been explored in [8]. In fact, perhaps the most transparent example of the dynamical nature of the weights w_Γ is the three-dimensional pure gravity description of the BTZ black hole [9]. As in four dimensions, the relevant physical states for the BTZ black hole are described as (quantum) spin networks that are eigenstates of the horizon length operator for length eigenvalues close to the macroscopic value L . However, one can argue that two eigenstates that have the same length eigenvalue L but which differ by the shape of the graph Γ , represent in fact one and the same physical state due to the absence of local degrees of freedom. Thus, considering a uniform Boltzmann weight w_Γ is ruled out in this case by dynamical considerations. This is a clear example where the dynamical nature of the weights w_Γ is manifest.

In this work, we leave aside the issue of finding the appropriate weights w_Γ , and concentrate instead on the question of whether it is possible to compute the number of microstates $N_\Gamma(a_H, \gamma)$ for a fixed Γ when the Barbero-Immirzi parameter is purely imaginary. More precisely, in a spirit similar to [10], we propose an expression of $N_\Gamma(a_H, \gamma)$ for $\gamma \in i\mathbb{R}$ based on the analytic continuation of Chern-Simons theory from a compact gauge group to the associated complex gauge group. The striking result is that the asymptotic behavior of the Chern-Simons Hilbert space dimension for $N_\Gamma(a_H, \pm i)$ when a_H is large, is in itself compatible with the Bekenstein-Hawking entropy, i.e.

$$\log(N_\Gamma(a_H, \pm i)) \stackrel{\text{s.c.}}{\sim} \frac{a_H}{4\ell_{\text{Pl}}^2}. \quad (3)$$

This result suggests that the complex formulation in terms of the Ashtekar variables (which is to be put in parallel with the so-called Lorentz-covariant formulation [11]) could lead to a clear-cut derivation of the black hole entropy in the framework of quantum gravity.

Analytic continuation to purely imaginary γ

Black holes in LQG can be described in terms of an $\text{SU}(2)$ Chern-Simons theory

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_\Delta \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (4)$$

where $k \propto a_H$ (the precise form of the level will not be important), and Δ is the black hole horizon with spatial area a_H . This area a_H is obtained as the sum of the fundamental contributions carried by the p links ℓ crossing the horizon, according to the formula

$$a_H = 8\pi\ell_{\text{Pl}}^2\gamma \sum_{\ell=1}^p \sqrt{C(j_\ell)}, \quad (5)$$

where $\sqrt{C(j_\ell)}$ is the quantum of area associated to the puncture $\ell \in \llbracket 1, p \rrbracket$, which is colored with an $\text{SU}(2)$ representation of spin j_ℓ and of finite dimension d_ℓ . Usually,

$C(j) = j(j+1)$ is the quadratic Casimir operator evaluated in the representation j , but there exists also a linear model with $C(j) = j^2$, which corresponds to a different regularization of the area operator. These two expressions obviously agree for large spins.

The Hilbert space associated to a black hole whose horizon is crossed by p colored links, is exactly the Hilbert space of the $\text{SU}(2)$ Chern-Simons theory (4) with p punctures colored with the representations of spins (j_1, \dots, j_p) . Its dimension is therefore given by the finite sum [12]

$$D_k(j_1, \dots, j_p) = \frac{2}{k+2} \sum_{d=1}^{k+1} \sin^2 \left(\frac{\pi d}{k+2} \right) \prod_{\ell=1}^p \frac{\sin \left(\frac{\pi d d_\ell}{k+2} \right)}{\sin \left(\frac{\pi d}{k+2} \right)}, \quad (6)$$

where $d_\ell = 2j_\ell + 1$ is the dimension of the representation j_ℓ .

A configuration for a black hole of horizon area a_H is an assignment of p spins (j_1, \dots, j_p) compatible with a_H in the sense that the relation (5) is satisfied. As a consequence of this restriction, the number of microstates for a given puncture configuration is a function of the Barbero-Immirzi parameter as well, and is given by

$$N_\Gamma(a_H, \gamma) = D_k(j_1, \dots, j_p). \quad (7)$$

The total number of microstates is given by a sum of the type (2) over the colored graphs Γ .

Let us now consider the analytic continuation to purely imaginary values of the Barbero-Immirzi parameter. More precisely, we take $\gamma = \pm i$. In this case [13] the level of the Chern-Simons theory becomes purely imaginary as well, i.e. $k \rightarrow \pm i\lambda$, and the expression for the dimension of the Chern-Simons theory Hilbert space becomes

$$N_\Gamma(a_H, \pm i) \equiv \frac{2}{\lambda} \sum_{d=1}^{\lambda} \sinh^2 \left(\frac{\pi d}{\lambda} \right) \prod_{\ell=1}^p \frac{\sinh \left(\frac{\pi d d_\ell}{\lambda} \right)}{\sinh \left(\frac{\pi d}{\lambda} \right)}, \quad (8)$$

where the sum runs from $d = 1$ to $d = \lambda$. When k is real and large, the sum runs from $d = 1$ to $d = k + 1 \sim k$. Here $k = i\lambda$, and we make the assumption¹ that the sum is still bounded and runs from $d = 1$ to $d = \lambda$. When the spins are large, this sum is dominated by the exponential

¹ The reason for keeping the sum bounded is clear from the mathematical point of view because it makes the sum convergent. The same trick has been considered in three dimensions to recover the BTZ black hole entropy [9].

with the largest argument (obtained for $d = \lambda$), which is given by

$$N_\Gamma(a_H, \pm i) \approx \frac{\alpha}{a_H} \prod_{\ell=1}^p \frac{\sinh(\pi d_\ell)}{\sinh(\pi)}, \quad (9)$$

where α is a constant independent of a_H . For a fixed graph Γ , all the spins become large in the large a_H limit. Thus, the entropy $S = \log(N_\Gamma(a_H, \pm i))$ of a given configuration behaves in this limit as

$$S \sim \frac{a_H}{4\ell_{\text{Pl}}^2}. \quad (10)$$

Discussion

To interpret this result, one should remember that the introduction of the Barbero-Immirzi parameter along with the real $\mathfrak{su}(2)$ Ashtekar-Barbero connection was in the first place a trick in order to avoid dealing with the reality conditions of the complex Ashtekar theory (which is recovered in the case $\gamma = \pm i$). Although this procedure is advantageous because it yields a Hamiltonian theory of gravity with first class constraints and with a compact gauge group, several important subtleties have been identified: i) the scalar constraint of the Ashtekar-Barbero theory is more complicated than in the self-dual case; ii) the Ashtekar-Barbero connection is not a spacetime connection [11, 14]; iii) the Barbero-Immirzi parameter appears in the quantum theory as a quantization ambiguity. These difficulties are absent in the (anti) self-dual formulation.

Our result suggests that a self-dual quantum theory might exist, and that it could lead to a clear-cut derivation of the Bekenstein-Hawking entropy formula. In fact, one can understand our computation as follows. First, one passes through a compact gauge group by means of the Ashtekar-Barbero connection. This unravels the whole machinery of LQG, and leads in particular to the notion of quantum geometry supported by spin network states, and to the Chern-Simons description of black holes. Then we perform an analytic continuation of physical quantities from $\gamma \in \mathbb{R}$ to $\gamma = \pm i$ as a way to recover the results associated with the self-dual theory.

The analytic continuation that we propose should be interpreted as an analytic continuation from the compact gauge group $\text{SU}(2)$ to the non-compact group $\text{SL}(2, \mathbb{C})$. If this is the case, then what is the meaning of the representations of finite dimension d_ℓ appearing in formula (9)? In fact, these are self-dual or anti self-dual representations of $\text{SL}(2, \mathbb{C})$, which indeed turn out to be finite-dimensional. To see this, let us recall that representations of $\text{SL}(2, \mathbb{C})$ are labelled by a pair (p, j) , where $p \in \mathbb{C}$ and $j \in \mathbb{N}/2$. The (anti) self-dual representations are those satisfying the constraint

$$\vec{L} \pm i\vec{K} = 0. \quad (11)$$

This condition is solved for $p = \pm i(j+1)$ [15]. This means that (anti) self-dual representations are finite-dimensional, and labelled by spins $j \in \mathbb{N}/2$ (in one-to-one correspondence with the $\text{SU}(2)$ spin networks that are the quantum states for real values of the Barbero-Immirzi parameter). Most importantly, the two Casimir operators, $C_1 \equiv J \cdot J = (p^2 - j^2 + 1)/2$ and $C_2 \equiv \star J \cdot J = pj$, are simply related by $C_1 = \pm iC_2 = -j(j+1)$. Therefore, for a suitable choice of inner product implementing the reality of the geometry, one can expect the area eigenvalues to coincide with those obtained for $\gamma = \pm 1$, as assumed here.

Notice as well that the (anti) self-dual representations $(\pm i(j+1), j)$ are exactly those appearing in the EPRL [16] model $(\gamma(j+1), j)$ if we take $\gamma = \pm i$. Since these representations are not unitary representations, they are not allowed in the usual construction of spin foam models or projected spin networks. However, they are simply related to the unitary representations in the sense that representation matrices in the canonical basis are analytic functions of p . Finally, the constraint $C_1 = \pm iC_2$ that is imposed for (anti) self-dual representations, and that is implicitly used in the analytic continuation of this letter, is known to be part of the reality constraints of the Ashtekar complex formulation. All this indicates that we might be on a way to defining a quantum theory for the Ashtekar variables [15].

Let us point out once again that the analytic continuation of Chern-Simons theory to a complex level k , when k is purely imaginary, is intimately related to trading the compact gauge group $\text{SU}(2)$ for $\text{SL}(2, \mathbb{C})$. The quantization of Chern-Simons theory with the Lorentz group naturally leads to the quantum Lorentz group $U_q(\mathfrak{sl}(2, \mathbb{C}))$ with real q , and the physical states are described in terms of the irreducible representations of $U_q(\mathfrak{sl}(2, \mathbb{C}))$. Similarly to the representations of the classical Lorentz group, representations of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ are labelled by a couple $\chi = (p, j)$ with $p \in \mathbb{C}$ and $j \in \mathbb{N}/2$. The (anti) self-duality constraint $p = \pm i(j+1)$ follows from the reality conditions as previously discussed. Moreover, the formula for the dimension of the $\text{SU}(2)$ Chern-Simons Hilbert space (6) is constructed using the so-called Verlinde coefficients S_{j_1, j_2} for any two representations j_1 and j_2 . An analogous formula for $\text{SL}(2, \mathbb{C})$ Chern-Simons theory should therefore involve the Verlinde coefficients S_{χ_1, χ_2} for the quantum Lorentz group, and it is easy to see that for $p = \pm i(j+1)$ in the representations $\chi_1 = (p_1, j_1)$ and $\chi_2 = (p_2, j_2)$ they become [17]

$$S_{\chi_1, \chi_2} \equiv S_{j_1, j_2} \equiv [(2j_1 + 1)(2j_2 + 1)], \quad (12)$$

where the q -numbers $[x]$ are defined with q real. More precisely, the coefficients correspond to the q -evaluation of the $\text{SU}(2)$ dimension, but with a real q instead of a root of unity. This is in fact what transforms the sin functions into sinh functions in (6). It can be seen that our procedure is indeed the natural way to define the analog of the

dimension of the Chern-Simons Hilbert space (in fact, the volume of the space of non-commutative connections) in the case of a non-compact gauge group [17].

Finally, let us stress that an important property of our construction is that the leading term in the black hole entropy is independent of the explicit form of the level of k . The only requirement is that k scale as a_H when the area becomes large.

Furthermore, the main result of this letter, which is the asymptotic behavior of the entropy for large a_H , does not depend on the exact form of the area spectrum. It is clear that the computation is a bit simpler when the spectrum is linear, but the result still holds when the quantum of area is the standard one of loop quantum gravity.

Many interesting questions are still open. These concern the sum over the graphs, the sub-leading (logarithmic) corrections to the entropy, the relationship with the approach of [18], the link with self-dual gravity, and the relation between our model and the computation of the entropy with a real γ . We hope to address some of these questions in the future.

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